Integrali indefiniti

Calcolo di integrali con il metodo di integrazione per parti

Ricordiamo che sussiste la seguente uguaglianza

$$\int f'(x)\cdot g(x)dx = f(x)\cdot g(x) - \int f(x)\cdot g'(x)dx$$

Esercizi proposti

1. $\int x^2\cdot\ln(x)dx = \frac{x^3}{9}\cdot(3\ln(x)-1)+c$
2. $\int\sqrt{x}\cdot\ln(x)dx = \frac{2}{3}\cdot x\sqrt{x}\cdot\left(\ln(x)-\frac{2}{3}\right)+c$
3. $\int\frac{\ln(x)}{\sqrt{x}}dx = 2\sqrt{x}\cdot(\ln(x)-2)+c$
4. $\int(4x^3-3)\cdot\ln(3x)dx = (x^4-3x)\cdot\ln(3x)-\frac{x^4}{4}+3x+c$
5. $\int(2x+1)\cdot\ln(x+1)dx = (x^2+x)\cdot\ln(x+1)-\frac{x^2}{2}+c$
6. $\int e^x\cdot\ln(e^x+1)dx = e^x\cdot\ln(e^x+1)-e^x+\ln(e^x+1)+c$ Applicazione dei metodi di integrazione per parti e per sostituzione.
7. $\int\text{arctg}(x)dx = x\cdot\text{arctg}(x)-\frac{1}{2}\ln(1+x^2)+c$
8. $\int\text{arc}\,sen(x)dx = x\cdot\text{arc}\,sen(x)+\sqrt{1-x^2}+c$
9. $\int x\cdot sen(x)dx = -x\cdot\cos(x)+sen(x)+c$
10. $\int x^2\cdot sen(x)dx = (2-x^2)\cdot\cos(x)+2x\cdot\text{sen}(x)+c$ Applicare due volte il metodo di integrazione per parti.
11. $\int x\cdot e^x dx = (x-1)\cdot e^x+c$
12. $\int 4x\cdot e^{2x}dx = (2x-1)e^{2x}+c$
13. $\int x\cdot e^{3x+4}dx = \frac{1}{3}\left(x-\frac{1}{3}\right)e^{3x+4}+c$
14. $\int(ax+b)e^x dx = (ax+b-a)\cdot e^x+c$, *con a e b costanti.*
15. $\int(x^2-x)\cdot e^{-x}dx = -(x^2+x+1)\cdot e^{-x}+c$
16. $\int(x^2-x)\cdot e^x dx = (x^2-3x+3)\cdot e^x+c$
17. $\int\frac{x+2}{e^x}dx = -(x+3)\cdot e^{-x}+c$
18. $\int\ln^2(x)dx = x\cdot\left[\ln^2(x)-2\ln(x)+2\right]+c$
19. $\int\sqrt{1-x^2}dx = \frac{1}{2}\cdot\left[x\sqrt{1-x^2}+arcsen(x)\right]+c$ Questo integrale si può risolvere anche per sostituzione ponendo $x=sen(t)$.
20. $\int\frac{x}{\sqrt{x+1}}dx = \frac{2}{3}(x-2)\sqrt{x+1}+c$

Elaborazioni

1. $\int x^2 \cdot \ln(x)dx = \int D\left(\frac{x^3}{3}\right) \cdot \ln(x)dx = \frac{x^3}{3} \cdot \ln(x) - \int \frac{x^3}{3} \cdot \frac{1}{x}dx = \frac{x^3}{3} \cdot \ln(x) - \frac{1}{3}\int x^2 dx = \frac{x^3}{3} \cdot \ln(x) - \frac{x^3}{9} + c$

2. $\int \sqrt{x} \cdot \ln(x)dx = \int D\left(\frac{2}{3} \cdot x^{\frac{3}{2}}\right) \cdot \ln(x)dx = \frac{2}{3} \cdot x^{\frac{3}{2}} \cdot \ln(x) - \int \frac{2}{3} \cdot x^{\frac{3}{2}} \cdot \frac{1}{x}dx = \frac{2}{3} \cdot x^{\frac{3}{2}} \cdot \ln(x) - \frac{2}{3} \cdot \int x^{\frac{1}{2}} dx$

   $= \frac{2}{3} \cdot x^{\frac{3}{2}} \cdot \ln(x) - \frac{4}{9} \cdot x^{\frac{3}{2}} + c = \frac{2}{3} \cdot x\sqrt{x} \cdot \left(\ln(x) - \frac{2}{3}\right) + c$

3. $\int \frac{\ln(x)}{\sqrt{x}}dx = 2\int \frac{\ln(x)}{2\sqrt{x}}dx = 2\int \ln(x) \cdot D\left(\sqrt{x}\right)dx = 2\left[\sqrt{x} \cdot \ln(x) - \int \sqrt{x} \cdot \frac{1}{x}dx\right] = 2\sqrt{x} \cdot \ln(x) - 2\int x^{-\frac{1}{2}}dx =$

   $2\sqrt{x} \cdot \ln(x) - 2 \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{x} \cdot (\ln(x) - 2) + c$

4. $\int \left(4x^3 - 3\right) \cdot \ln(3x)dx = \int D\left(x^4 - 3x\right) \cdot \ln(3x)dx = \left(x^4 - 3x\right) \cdot \ln(3x) - \int \left(x^4 - 3x\right) \cdot \frac{1}{3x} \cdot 3dx$

   $= \left(x^4 - 3x\right) \cdot \ln(3x) - \int \left(x^3 - 3\right)dx = \left(x^4 - 3x\right) \cdot \ln(3x) - \frac{x^4}{4} + 3x + c$

5. $\int (2x+1) \cdot \ln(x+1)dx = \int D\left(x^2 + x\right) \cdot \ln(x+1)dx = \left(x^2 + x\right) \cdot \ln(x+1) - \int x\cancel{(x+1)} \cdot \frac{1}{\cancel{(x+1)}}dx =$

   $\left(x^2 + x\right) \cdot \ln(x+1) - \frac{x^2}{2} + c$

6. $\int e^x \cdot \ln\left(e^x + 1\right)dx = \int D\left(e^x\right) \cdot \ln\left(e^x + 1\right)dx = e^x \cdot \ln\left(e^x + 1\right) - \int \frac{e^{2x}}{e^x + 1}dx$. Per l'integrale residuo applichiamo il metodo di integrazione per sostituzione ponendo $e^x = t \rightarrow x = \ln(t)$, e $dx = \frac{1}{t}dt$. Si ha:

   $-\int \frac{e^{2x}}{e^x + 1}dx = -\int \frac{t^2}{t+1} \cdot \frac{1}{t}dt = -\int \frac{t+1-1}{t+1}dt = -\int \left(1 - \frac{1}{t+1}\right)dt = -t + \ln\left(|t+1|\right) + c = -e^x + \ln\left(e^x + 1\right) + c$.

   In definitiva risulta

   $\int e^x \cdot \ln\left(e^x + 1\right)dx = e^x \cdot \ln\left(e^x + 1\right) - e^x + \ln\left(e^x + 1\right) + c$

7. $\int \text{arctg}(x)dx = \int 1 \cdot \text{arctg}(x)dx = \int D(x) \cdot \text{arctg}(x)dx = x \cdot \text{arctg}(x) - \int x \cdot \frac{1}{1+x^2}dx = x \cdot \text{arctg}(x) - \frac{1}{2}\int \frac{2x}{1+x^2}dx =$

   $x \cdot \text{arctg}(x) - \frac{1}{2}\ln\left(1 + x^2\right) + c$

8. $\int \text{arc}\,sen(x)dx = \int 1 \cdot \text{arc}\,sen(x)dx = \int D(x) \cdot \text{arc}\,sen(x)dx = x \cdot \text{arc}\,sen(x) - \int x \cdot \frac{1}{\sqrt{1-x^2}}dx =$

   $x \cdot \text{arc}\,sen(x) - \int \left(1 - x^2\right)^{-\frac{1}{2}} \cdot xdx = x \cdot \text{arc}\,sen(x) + \frac{1}{2}\int \left(1 - x^2\right)^{-\frac{1}{2}} \cdot (-2x)dx = x \cdot \text{arc}\,sen(x) + \frac{1}{2} \cdot \frac{\left(1 - x^2\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c =$

   $x \cdot \text{arc}\,sen(x) + \sqrt{1 - x^2} + c$

9. $\int x \cdot sen(x)dx = \int x \cdot D(-\cos(x))dx = -x \cdot \cos(x) - \int -\cos(x) \cdot 1dx = -x \cdot \cos(x) + sen(x) + c$

10. $\int x^2 \cdot sen(x)dx = \int x^2 \cdot D(-\cos(x))dx = -x^2 \cdot \cos(x) + \int 2x \cdot \cos(x) \cdot dx = -x^2 \cdot \cos(x) + 2\int x \cdot D(\text{sen}(x)) \cdot dx =$

    $-x^2 \cdot \cos(x) + 2\left[x \cdot \text{sen}(x) - \int 1 \cdot \text{sen}(x)dx\right] = -x^2 \cdot \cos(x) + 2x \cdot \text{sen}(x) + 2\cos(x) + c = \left(2 - x^2\right) \cdot \cos(x) + 2x \cdot \text{sen}(x) + c$

11. $\int x\cdot e^x dx=\int x\cdot D\left(e^x\right)dx= x\cdot e^x-\int 1\cdot e^x dx= x\cdot e^x-e^x+c=(x-1)\cdot e^x+c$

12. $\int 4x\cdot e^{2x}dx=2\int x\cdot D\left(e^{2x}\right)dx=2\left[x\cdot e^{2x}-\int 1\cdot e^{2x}dx\right]=2x\cdot e^{2x}-\int 2e^{2x}dx=2x\cdot e^{2x}-e^{2x}+c=(2x-1)e^{2x}+c$

13. $\int x\cdot e^{3x+4}dx=e^4\cdot\frac{1}{3}\int x\cdot 3e^{3x}dx=\frac{e^4}{3}\int x\cdot D\left(e^{3x}\right)dx=\frac{e^4}{3}\left[xe^{3x}-\int 1\cdot e^{3x}dx\right]=\frac{e^4}{3}\left[xe^{3x}-\frac{1}{3}e^{3x}\right]+c=\frac{1}{3}\left(x-\frac{1}{3}\right)e^{3x+4}+c$

14. $\int (ax+b)e^x dx=\int (ax+b)\cdot D\left(e^x\right)dx=(ax+b)\cdot e^x-\int a\cdot e^x dx=(ax+b)\cdot e^x-ae^x+c=(ax+b-a)\cdot e^x+c$

15. $\int \left(x^2-x\right)\cdot e^{-x}dx=\int\left(x^2-x\right)\cdot D\left(-e^{-x}\right)dx=\left(x^2-x\right)\cdot\left(-e^{-x}\right)-\int(2x-1)\cdot\left(-e^{-x}\right)dx=$

$\left(x^2-x\right)\cdot\left(-e^{-x}\right)-\int(2x-1)\cdot D\left(e^{-x}\right)dx=\left(x^2-x\right)\cdot\left(-e^{-x}\right)-\left[(2x-1)\cdot e^{-x}-\int 2\cdot e^{-x}dx\right]=$

$\left(x^2-x\right)\cdot\left(-e^{-x}\right)-(2x-1)\cdot e^{-x}+2\int e^{-x}dx=\left(x^2-x\right)\cdot\left(-e^{-x}\right)-(2x-1)\cdot e^{-x}-2e^{-x}+c=-\left(x^2+x+1\right)\cdot e^{-x}+c$

16. $\int\left(x^2-x\right)\cdot e^x dx=\left(x^2-3x+3\right)\cdot e^x+c$ Esercizio a cura del Lettore

17. $\int\frac{x+2}{e^x}dx=\int(x+2)\cdot e^{-x}dx=\int(x+2)\cdot D\left(-e^{-x}\right)dx=(x+2)\left(-e^{-x}\right)-\int 1\cdot\left(-e^{-x}\right)dx=\ldots=-(x+3)\cdot e^{-x}+c$

18. $\int \ln^2(x)dx=\int\ln^2(x)\cdot D(x)dx=x\cdot\ln^2(x)-\int x\cdot 2\ln(x)\cdot\frac{1}{x}dx=x\cdot\ln^2(x)-2\int\ln(x)\cdot 1dx=$

$x\cdot\ln^2(x)-2\left[x\cdot\ln(x)-\int x\cdot\frac{1}{x}dx\right]=x\cdot\ln^2(x)-2x\cdot\ln(x)+2x+c=x\cdot\left[\ln^2(x)-2\ln(x)+2\right]+c$

19. $\int\sqrt{1-x^2}dx=\int\sqrt{1-x^2}\cdot D(x)dx=x\sqrt{1-x^2}-\int x\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)dx=x\sqrt{1-x^2}-\int\frac{-x^2}{\sqrt{1-x^2}}dx=$

$x\sqrt{1-x^2}-\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx=x\sqrt{1-x^2}-\left[\int\frac{1-x^2}{\sqrt{1-x^2}}dx-\int\frac{1}{\sqrt{1-x^2}}dx\right]=x\sqrt{1-x^2}-\left[\int\sqrt{1-x^2}dx-arcsen(x)\right]=$

$x\sqrt{1-x^2}-\int\sqrt{1-x^2}dx+arcsen(x)$, trasportando al primo membro l'integrale residuo e dividendo i due membri per 2 si ottiene

$\int\sqrt{1-x^2}dx=\frac{1}{2}\cdot\left[x\sqrt{1-x^2}+arcsen(x)\right]+c$

20. $\int\frac{x}{\sqrt{x+1}}dx=2\int\frac{x}{2\sqrt{x+1}}dx=2\int x\cdot D\left(\sqrt{x+1}\right)dx=2\left[x\sqrt{x+1}-\int 1\cdot\sqrt{x+1}dx\right]=2x\sqrt{x+1}-2\int(x+1)^{\frac{1}{2}}dx=$

$2x\sqrt{x+1}-2\cdot\frac{(x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c=2x\sqrt{x+1}-2\cdot\frac{2}{3}(x+1)\cdot\sqrt{x+1}+c=\frac{2}{3}(x-2)\sqrt{x+1}+c$